PROBLEM SET 12

JIAHAO HU

Problem 1. *Prove the following.*

(1) *If* $f = \sum_1^n c_j \chi_{[a_j, b_j)}$ *is a step function, define*

$$I_f(\omega) = \int_0^\infty f(t) d\omega(t) = \sum_1^n c_j \left[\omega(b_j) - \omega(a_j)\right].$$

Then I_f *is an* L^2 *random variable on* Ω_c *with mean 0 and variance* $\|f\|_2^2 = \int |f|^2 dx$.

(2) *The map* $f \to I_f$ *extends to an isometry from* $L^2([0,\infty))$ *to* $L^2(\Omega_c)$.

(3) *If* $f \in BV([0,\infty))$ *is right continuous and* $\operatorname{supp}(f)$ *is compact, there is a sequence* $\{f_n\}$ *of step functions such that* $f_n \to f$ *in* L^2 *and* $df_n \to df$ *vaguely.*

(4) *If* $f \in BV([0,\infty))$ *is right continuous and* $\operatorname{supp}(f)$ *is compact, then* $I_f(\omega) = -\int_0^\infty \omega(t) df(t)$ *almost surely.*

Proof. Let $\{X_t\}_{t\geq 0}$ denote the Wiener process.

(1) Without loss of generality, we may assume intervals $[a_j, b_j)$'s are disjoint. Then notice that $I_f = \sum_1^n c_j (X_{b_j} - X_{a_j})$. Since $(X_{b_j} - X_{a_j})$'s are independent L^2 random variables with mean 0 and variance $(b_j - a_j)$, I_f is an L^2 random variable with mean $\sum_1^n c_j \cdot 0 = 0$ and variance $\sum_1^n c_j^2 (b_j - a_j) = \|f\|_2^2$.

(2) From (1) we see if f is a step function, then since I_f has mean 0 we have $\|I_f\|^2_{L^2(\Omega_c)} = \sigma^2(I_f) = \|f\|^2_{L^2([0,\infty))}$. By density of step functions in $L^2([0,\infty))$, $f \mapsto I_f$ uniquely extends to an isometry from $L^2([0,\infty))$ to $L^2(\Omega_c)$.

(3) By dealing with positive and negative variations separately, we may assume f is increasing. Now f is of bounded variation, right continuous and of compact support, $df \in M([0,\infty))$ is a finite positive Borel measure. Then by Problem 5 in Homework 9, there exists measures μ_n supported on finite sets such that $\mu_n \to df$ vaguely, moreover we can take μ_n to be positive. Consider $f_n(x) := \mu_n((0,x]) + f(0)$, then since μ_n is supported on finite points we have f_n is a step function, and $df_n \to df$ vaguely by definition. We claim $f_n \to f$ in L^2. Indeed, f is continuous a.e. so by Prop. 7.19 $f_n \to f$ a.e., then $f_n \to f$ in L^2 by dominated convergence theorem.

(4) First for $f = \sum_1^n c_j \chi_{[a_j,b_j)}$, $df = \sum_1^n c_j (\delta_{a_j} - \delta_{b_j})$ and thus

$$-\int \omega(t) df(t) = -\sum_1^n c_j (\omega(a_j) - \omega(b_j)) = I_f(\omega).$$

In general, by (3) we can find step functions f_n such that $f_n \to f$ in L^2 with $df_n \to df$ vaguely. By (2) $I_{f_n} \to I_f$ in $L^2(\Omega_c)$, and replacing f_n by a subsequence if necessary we may assume $I_{f_n} \to I_f$ almost surely. Therefore, almost surely we have

$$I_f(\omega) = \lim_{n\to\infty} I_{f_n}(\omega) = \lim_{n\to\infty} -\int \omega(t)df_n(t) = -\int \omega(t)df(t).$$

□

Problem 2. *Let R_x and L_x denote right and left translation by x in a locally compact group G. Let μ be a Radon measure on G, and $f \in C_c(G)$. Show that the functions $x \to \int L_x f d\mu$ and $x \to \int R_x f d\mu$ are continuous.*

Proof. We shall prove $x \mapsto \int L_x f d\mu$ is continuous, similar argument works for $x \mapsto \int R_x f d\mu$. Since G is locally compact, we can find V_0 open and K compact such that $e \in V_0 \subset K$. Let $\epsilon > 0$, by Prop. 11.2 f is left uniformly continuous, so there is an open neighborhood $V \subset V_0$ of e such that $\|L_y f - f\|_u \le \epsilon/M$ where $M = \mu((x\cdot \operatorname{supp} f)\cup(xK\cdot \operatorname{supp} f))$. Note since $(x\cdot \operatorname{supp} f)\cup(xK\cdot \operatorname{supp} f)$ is compact and μ is Radon, we have $M < \infty$. Let $U = xV$, then for any $x' = xy \in U$ we have,

$$\begin{aligned}|\int L_x f d\mu - \int L_{x'} f d\mu| &\le \|L_x f - L_{xy} f\|_u \cdot \mu((x\cdot \operatorname{supp} f)\cup(xK\cdot \operatorname{supp} f)) \\ &= \|f - L_y f\|_u \cdot M \\ &\le \epsilon.\end{aligned}$$

This proves $x \to \int L_x f d\mu$ is continuous. □

Problem 3. *Let G be a locally compact group which is homeomorphic to an open subset U of $\mathbb{R}^n$ in such a way that, if we identify G with U, left translation is an affine map – that is, $xy = A_x(y)+b_x$ where A_x is a linear transformation of $\mathbb{R}^n$ and $b_x \in \mathbb{R}^n$. Show that $|\det A_x|^{-1}dx$ is a left Haar measure on G, where dx denotes Lebesgue measure on $\mathbb{R}^n$.*

Proof. We notice for $x_1, x_2 \in G$, $x_1x_2y = A_{x_1}A_{x_2}(y) + (A_{x_1}b_{x_2} + b_{x_1})$ for all $y \in G$, so $A_{x_1x_2} = A_{x_1}A_{x_2}$. In particular $A_{x^{-1}} = (A_x)^{-1}$, hence $d\mu(x) = |\det A_x|^{-1}dx$ is nonzero. It is clear A_x is continuous with respect to x, so μ is Radon.

Let $y' = x^{-1}y = (A_x)^{-1}(y) + b_{x^{-1}}$, then $dy' = |\det A_x|^{-1}dy$, thus

$$d\mu(y') = |\det A_{y'}|^{-1}dy' = |\det A_{y'}|^{-1}|\det A_x|^{-1}dy = |\det A_y|^{-1}dy = d\mu(y).$$

Therefore for any $f \in C_c^+$ we have

$$\int L_x f d\mu = \int L_x f(y)d\mu(y) = \int f(y')d\mu(y) = \int f(y')d\mu(y') = \int f d\mu.$$

By Prop. 11.4, μ is is a left Haar measure. □

Problem 4. *Let $G = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x > 0$ and $y \in \mathbb{R}$. Construct a Borel set in G with finite left Haar measure but infinite right Haar measure. Construct a left uniformly continuous function on G that is not right uniformly continuous.*

Proof. We identify G with $\mathbb{R}_+ \times \mathbb{R} \subset \mathbb{R}^2$. Then under this identification $(x, y)\cdot(z, w) = (xz, xw + y) = (xI_2)(z, w) + (0, y)$ where I_2 is the identity linear transformation on $\mathbb{R}^2$. Thus by Problem 3, $|\det(xI_2)|^{-1}dxdy = x^{-2}dxdy$ is a left Haar measure on G. Similarly $(z, w)\cdot(x, y) = \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}(z, w)$, so $x^{-1}dxdy$ is a right Haar measure on G.

(1) Let $E = [1,\infty) \times [0,1] \subset G$, then it has measure 1 with respect to left Haar measure $x^{-2}dxdy$ and infinite right Haar measure.

(2) Let $f(x,y) = \sin y$. Then for $\sigma = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} \in G$ where u is close to 1 and v close to 0, we have $|L_{\sigma^{-1}} f(x,y) - f(x,y)| = |(u-1)\sin y + v| \leq |u-1| + |v|$. This shows f is left uniformly continuous. However, we claim $\limsup_{\sigma \to e} \|R_\sigma f - f\|_u \geq 1$. Indeed, for any $v > 0$

$$\|R_\sigma f - f\|_u \geq |R_\sigma f(\pi/2v, 0) - f(\pi/2v, 0)| = 1.$$

□

Problem 5. *Let $\{G_\alpha\}_{\alpha \in A}$ be a family of topological groups and $G = \prod_{\alpha \in A} G_\alpha$. Prove that with product topology and coordinatewise multiplication, G is a topological group. If each G_α is compact and μ_α is the Haar measure on G_α such that $\mu_\alpha(G_\alpha) = 1$, then the Radon product of the $\mu_\alpha' s$ is a Haar measure on G.*

Proof. (1) Let $g = \prod_{\alpha \in A} g_\alpha \in G$. Since both $\pi_\alpha : G \to G_\alpha, L_{g_\alpha}$ are continuous, so is $\pi_\alpha \circ L_g = L_{g_\alpha} \circ \pi_\alpha$ for all $\alpha \in A$. This proves L_g is continuous. Similarly one can prove inverse is continuous. Therefore G is a topological group.

(2) By Tychnoff's theorem, G is compact Hausdorff, in particular locally compact. By Prop. 11.4, it suffices to show $\int f d\mu = \int L_y f d\mu$ for all $f \in C_c^+(G) \subset C(G)$. Moreover, since $C_F(G)$ is dense in $C(G)$ and μ Radon with $\|\mu\| = \mu(G) < \infty$, it suffices to check for $f \in C_F(G)$. Then we may regard f as a function on a finite product of G_α's, the rest will follow easily from the invariance of Haar measure μ_α's and Fubini's theorem.

□